Universidad de Guantánamo

Ciencia y Progreso
Publicada en línea: enero 2021, RNPS: 2477, ISSN: 2707-7098
http://cienciayprogreso.cug.co.cu

El reto de enseñar a demostrar Geometría
The challenge of teaching to demonstrate Geometry

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Recibido: 25 de marzo de 2020
Aceptado: 26 de noviembre de 2020

Resumen
Aprender a demostrar proposiciones matemáticas es difícil para el estudiante, como para el profesor de Matemática enseñar las demostraciones, es una realidad de la praxis. Pero, ¿por qué? En el proceso de enseñanza-aprendizaje de la Matemática, están dadas las condiciones para garantizar el éxito en este empeño, ¿o no? Este trabajo aborda algunas aristas que se observan en todo este proceso, demostrar, enseñar a demostrar en la escuela, y propone una salida didáctica, que si bien no pretende agotar el tema, procura esclarecer el actuar de los docentes al enfrentar esta tarea en el caso concreto de la Geometría.

Palabras clave: Geometría; Demostración en geometría; Enseñanza de la geometría; Enseñanza-aprendizaje de la Matemática

Abstract
Learning to demonstrate mathematical propositions is difficult for the student, as it is for the mathematics teacher to teach the demonstrations; it is a reality of practice. But why? In the teaching-learning process of Mathematics, the conditions are given to guarantee success in this endeavor, or not? This work deals with some aspects that can be observed in this whole process, demonstrating, teaching to demonstrate at school, and proposes a didactic solution, which, although it does not pretend to exhaust the subject, tries to clarify the teachers' actions when facing this task in the specific case of Geometry.

Keywords: Geometry; Demonstration in Geometry; Teaching of Geometry; Teaching-learning of Mathematics

Introduction

Today it is a necessity for the formation of the student, to conceive, to form, to develop and to evaluate his competences in the before, the during and the after at the time of solving any exercise of mathematical demonstration, that to the effects of this article, and following the canons of the Didactics of the current Mathematics, we will call mathematical propositions. A justification of this need is based on the possibilities that one has, when demonstrating mathematical propositions, of formalizing a thought as formal as possible, with the advantages that this implies.

And this is true: demonstrating in Mathematics contributes to create a concrete thought, but quite formal. The question now is not only to determine how much formality is needed in a student at the end of the intermediate level, but also, in who determines that formality. The second question is clear: it is more of a curricular issue; it depends on the program, the methodological orientations, etc. The first element is not so clear. In fact, in the process of learning to demonstrate by the student, and of teaching to demonstrate by the mathematics teacher, there are classroom situations that describe the teacher's didactic solutions to his sometimes unjustified incompetence to demonstrate and teach to demonstrate in mathematics, which are summarized below without further comment.

Excessive use of redundancies, repetitions, comparisons and other analogies that supposedly make the demonstration procedures memorize.

Explanations, representations, exemplifications and comments through which he will try to link the knowledge of the students to that of his axiomatized mathematical exposition.

Use, at times, of inappropriate didactic resources aimed at combating one of his many inappropriate classroom tendencies:

- Speaking: what prevents the student from doing it himself?

- Making the student talk instead of making him act.
- Teaching instead of letting the student develop according to his spontaneous development.
- To build knowledge according to the teacher's culture instead of letting the student build his knowledge in all creativity.
- To concentrate the student's attention on himself instead of returning him to the beneficial

influence of small groups working freely.

Development

These questions explain the rise in recent years of the tendency, from the perspective of Didactics of Mathematics, to formulate, demonstrate and apply theorems, but outside the formalism demanded by the science of Geometry, which is well justified, since the process of mathematical demonstration, in classroom conditions, must be considered a unique case of mathematical argumentation of the veracity of the propositions (expressed in theorems or demonstration exercises).

But reality shows that our students still do not know how to demonstrate mathematical propositions, and that the teachers of the Discipline have limitations in teaching the corresponding procedures, which justifies, however, the efforts of clarification made in the field about the very notion of mathematical proof, its different typologies and mutual relations are insufficient. Today, the idea of demonstration prevails, understood in a rigid and absolute way within the mathematical community, and it seems to be the only possible and worthy conception to be taken into account.

Precisely, this work tries to analyze the differences of meaning of the idea of demonstration, passing and clarifying concepts such as mathematical demonstration, conjecture in classroom situations, approaching, in addition, some edges that are observed in all this process of demonstrating and teaching to demonstrate in the school as Ballester (1992), Godino (1997) and Beltrán (2015) have studied, and proposes a didactic exit, that although it does not try to exhaust the subject, it tries to clarify the teachers' action when facing this task in the concrete case of Geometry.

Some fallacies in the process of mathematical demonstration

1. Examples and Counter-Examples

This is an important case of elements taken for a mathematical demonstration from examples that verify the compliance of the property.

The 5th year students of the Mathematics and Physics career were asked: How many examples will be necessary to demonstrate that the sum of the interior angles of a triangle is equal to 180º?

The answers were several, ranging from three to many. The common reasoning is that if it works for one, it works for all. This fallacy is recognized by Alvarez (2011). The next question to these same students was: If for one it doesn't work?

The way out of situations of this type in the classroom is the demonstration of false propositions from some counterexample.

In the previous example, the proposed proposition is true, and is recognized by all as a theorem. The issue is less evident when dealing with conjectures, or propositions not declared as a theorem.

An example of this type of situation that can be dealt with in math classes might be the polynomial p(n) = n2 + n + 41.

n	1	2	3	4	5	6	7	...	40
$n^2 + n + 41$	43	47	53	61	71	83	97	...	$1681 = 41^2$
			They are all prime numbers!						**¿?**

As you can see, for the numbers n = 40 and n = 41, the results obtained are compound numbers. The fact that for n = 42, the given polynomial is again a prime number should be discussed with the students.

Intuition, induction and deduction

Intuition is a basic concept of the Theory of Knowledge and applied in epistemology it is described as that knowledge that is direct and immediate, without the intervention of deduction or reasoning, being usually considered as evident. A first problem with this is: what is evident, how to know that something is true for being evident?

On the other hand, intuition analyzed from the perspective of logic does not necessarily lead to a false or fallacious result, although the mode of reasoning with which it has been arrived at may be. Not knowing the mode of reasoning makes us enter the fallacy ad ignorantiam (cognitive prejudice) preventing us from knowing if the method that our brain performs unconsciously is correct or not.

This is so, since knowing how the result has been reached allows the systematization, automation and verification of the reasoning pattern. Once this pattern is known, it can be tested and verified if it is true for all cases. The verification is very important because it allows to avoid self-deception and to approach the truth. In this way, one pases from intuitions to deduction.

Intuition is synthetic understanding; it is the first indication of a deep subjective unification. The most useful way to develop intuition is through the interpretation of symbols. Sometimes, the propaedeutic character of primary education confuses teachers and students in the demonstration of properties. It is the criterion of many that in these ages' demonstrations should not be rigorous. And they are not right. The formal construction of mathematics cannot be based on intuition. Our texts are made up of many errors. An example, Mathematics 3, p. 146: Point out the figures that have two equal sides

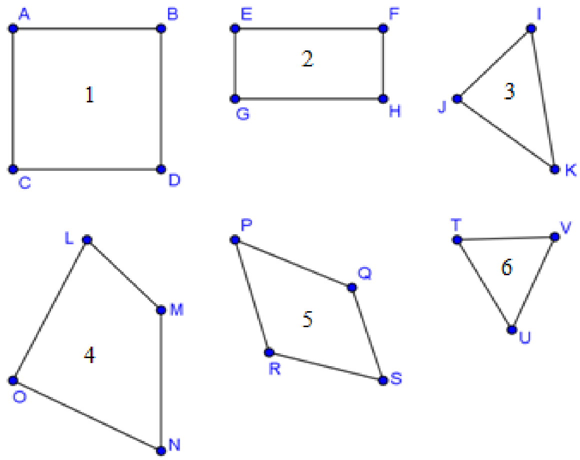


Figure 1. Exercise taken from Math Textbook 3, page 146.

Although the role played by intuition in the formation of the first mathematical concepts is not denied, as part of the sensoperceptible, representative, and even rational levels of the instrumental cognitive activity of the personality, abusing it or incurring in some non-rigorous practices, it leaves much to be said about mathematical activity and its deductive character that is sometimes developed.

A more favorable situation is achieved through induction, but without the excesses that are sometimes committed in mathematics classes.

Inductive reasoning is the type of reasoning in which the truth of the premises gives support to the truth of the conclusion, but does not guarantee it. Just this last part of the statement is the most important, and the one that costs more work in the Mathematics classes. The fact that the truth in question is not guaranteed, is the invitation to demonstrate it.

Rarely in the classes of Mathematics it is observed the demonstration of the theorem that refers that the sum of the amplitudes of the interior angles of a triangle is equal to 180º; rarer still is to demonstrate that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of its hypotenuse (Pythagorean Theorem). The same 5th graders were asked: Enunciate and prove one of the theorems you prove in your school math classes. Only one of them could remember a theorem, and could not prove it. On another occasion they were asked to enunciate and prove the Pythagorean Theorem. Here the mistakes were two: they do not remember with the rigor that is demanded by the statement, and none is able to at least start a demonstration of it.

One reason for this situation is that in most cases, an exaggerated induction of the concept or property is appealed to, and in very few cases the result in question is demonstrated.

The authors of this research do not believe that the absolute axiomatization of science or its teaching as premises to ensure its deductive character is relevant. An axiomatic system is the finished form that a deductive theory takes today. It is a system where all the undefined terms or objects and the unproven propositions are explicitly stated, the latter being fixed as hypotheses from which the other propositions of the system can be constructed, following perfectly and expressly determined logical rules. The logical chaining of the hypotheses constitutes the demonstration. The need for undefined terms and unproven propositions is due to the fact that it is impossible to carry out definition and demonstration indefinitely.

This is a good point. It is not intended that everything be demonstrated. It is that this is truly impossible, even more so within the scope of a study program. In any case, it is not correct not to demonstrate anything either, and to leave everything to intuition or to the use of purely inductive methods. By means of the demonstration, new propositions or relations between the objects are established from the relations given as axioms; then it becomes necessary to name or define the new objects that verify these properties.

The demonstrative character of the graph or figure

A problem that is presented today in the Mathematics classes, no matter what level it is, lies in the value of truth that is associated with the figure, as a demonstrative argument of certain property by teachers and students of Mathematics.

One example is enough: ABCD is a parallelogram, and E is a point on the AD side. Classify the BCE triangle according to its sides.

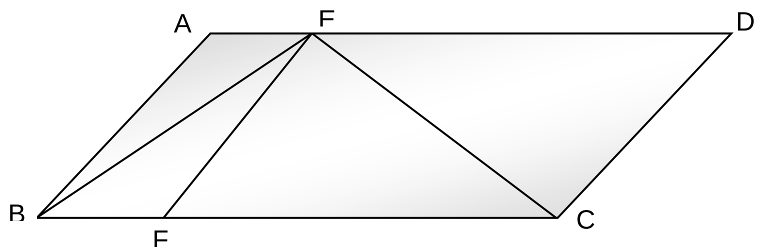


Figure 2. Figure classification exercise, which involves a demonstration of the property.

Under certain conditions, which include that the EF segment is parallel to the AB segment, the BE and EC segments are equal, so the given triangle isosceles of base BC, and not scalene as it seems. It is only added to this topic that the demonstrative character of the figure is null. Although it is a support to the demonstration of any property, to conclude that this is valid, as shown in the figure is false.

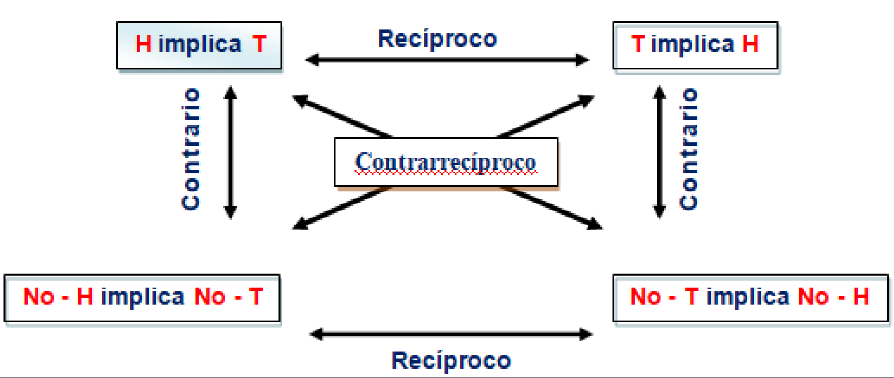
What to do then?

A. First thing: To equip oneself with a theory.

A first proposal can be observed from studying fragments of the text, Mathematical Analysis. Volume I by J. R. Pastor.

Given the theorem "H P" there are several theorems related to it. Two are called reciprocal, when each one has as its hypothesis the thesis of the other; those that have as their hypothesis and thesis respectively contrary propositions (H and not - H, T and not - T) are called opposites.

Here we indicate the given theorem (called direct) and also the reciprocal, the opposite and the reciprocal of the opposite (or opposite of the reciprocal), which we will call counter-reciprocal.



Figure,3. Relations between the direct theorem and its components: thesis and hypothesis.

A very common error in mathematics classes, and unfortunately almost at any level of teaching, is to declare or assume as valid the reciprocal of any theorem without previous demonstration. This is manifested by the ignorance of the following statement: The validity of a theorem does not imply that of its reciprocal, as can be seen in the diagrams of figure 1, in which the circles represent the extensions of the sets defined by propositions H and T.

The direct theorem expresses that H is sufficient for T to be fulfilled; the reciprocal says that H is necessary for T to be fulfilled. If both the direct and reciprocal theorems are valid, then H and T are equivalent (H = T), and H is said to be a necessary and sufficient condition for T to be fulfilled. This is the case in the example:

H: "The ABC triangle is equilateral"; T: "The ABC triangle is equiangular".

The method of reasoning that consists in proving the direct theorem by using the counter-reciprocal is called the absurd by reduction; that is, in order to prove that "H implies T", the falsity of T (non-T) is accepted, and from it the falsity of H (non-H) is deduced.

A detail is insisted on: the mathematical propositions are demonstrated in deductive form. No mathematical result can be considered valid, if it has not been deduced from the starting propositions. Familiarization with the mechanisms of proof used in mathematics constitutes a basic aspect for understanding a mathematical proof. Two common deductive situations in the mathematical activity are the search and the analysis of the proof of a proposition.

B. The second thing: To equip oneself with a procedure.

Polya (1989), makes considerations about the classification of the problems that the students face in the teaching-learning process of the subject Mathematics, because of the importance of the subject, we highlight its classification below. According to this author, it should be considered that the problems are divided into problems to be solved and problems to be demonstrated. In the problems to be solved, the purpose is to discover a certain object, the unknown of the problem, that is, what is sought or what is asked.

It also offers a list of questions and suggestions that guide the solution of the problems to be solved and the problems to be demonstrated, which are described later in this material.

In the problems to be demonstrated, the purpose is to show conclusively the accuracy or falsity of a clearly stated statement.

In this section, the teacher must take into account the following elements that govern his or her way of acting when confronting students with the solution of exercises. These, according to the authors of this paper, are 6, namely

1. Reformulate the proposal in an implicit manner: if..., then...

This is guaranteed by the "simple" recognition of the premise and the thesis, on which the Mathematics teacher must work with his or her students. Language translation exercises must be continuously approached with the students, and each one of these demonstration exercises must be treated with great care, until they form the corresponding skills to read an exercise to be demonstrated as a theorem.

II. Determining the type of demonstration

- Direct demonstration: We start from the premise of the proposition to be demonstrated, apply logical inference rules, and obtain the thesis.

Table,1. Didactic scheme for a direct demonstration.

<table>
<tr><td colspan="2">Theorem A $\rightarrow$ B</td></tr>
<tr><td>Premise: A</td><td>Thesis: B</td></tr>
<tr><td colspan="2">Demonstration

We assume that A is valid

We infer using theorems, definitions, logical rules, until:

We obtain B

Then it turns out that

if A then B</td></tr>
</table>

- **Indirect demonstration:** The logical negation of the proposition to be demonstrated is true, and it is verified that this assumption leads to a contradiction.

Table, 2. Didactic scheme for an indirect demonstration.

Theorem A $\rightarrow$ B	
Premise: A	Thesis: B
Demonstration We assume that it is not true that A B This is equivalent to assuming that A is true and B is not. We infer using theorems, definitions and logical rules, until: A contradiction arises The contradiction indicates that our assumption is false, then it turns out that If A, then B	

III. Searching for an idea and a demonstration plan

The search for the demonstration begins with the motivation of your need. Some of the arguments, to arouse students' interest in proving the veracity of a given proposition, are based on:

- Measurement error.
- Optical aberrations.
- False reciprocal demands of true propositions.
- Formulation of a universal proposition, after analyzing a limited number of cases.

Then attention is turned to the analysis of what is to be demonstrated. The first step is to understand the statement of the proposition. Some of the actions that contribute to this can be

- Is the proposal formulated in an implicit way?
- What are its premises?
- What is the thesis?
- Can you build a figure of analysis?
- Will it be necessary to reformulate the proposal?

- Likewise, it is important to specify the field of search:
- In what unit of matter is the proposal located?
- What contents were studied previously?
- Do you know the definitions of the concepts mentioned in the proposal?

Have other proposals with similar premises and theses been studied?

IV. Selection of special demonstration procedures

- Demonstration by construction of examples or counter-examples

Suppose you want to refute the following proposition:

If two chords of a circle are equal, then they are parallel.

It is enough to find a counterexample: Two different diameters, are two equal strings, however, they are cut.

- Demonstrations through counterexample

Take, for example, the proposition: If ABCD is a parallelogram, then its opposite sides are parallel

Reciprocal

If the opposite sides of an ABCD quadrilateral are parallel, then ABCD is a parallelogram

Counterpart

If the opposite sides of the quadrilateral ABCD are not parallel, then ABCD is not parallelogram

By directly demonstrating the counter-reciprocal, the validity of the original proposal is proven. Starting from the hypothesis, if the quadrilateral ABCD does not have the opposite sides parallel, by definition it is a trapezoid, which proves that ABCD is not a parallelogram.

V. Understandable demonstration display

To solve the problem of representation, we can rely on a succession of indications such as the following (in the case of direct demonstrations)

- Write the premises.
- Write the thesis.

- Draw a figure of analysis if necessary.
- -Write in an appropriate order all the conclusions that lead from the premise to the thesis. The thesis should appear at the end.
- -Verify the inferences and write the necessary foundations.

VI. Evaluation of the demonstration.

In the last phase of the representation of the demonstration, its outcome should be evaluated. Among others, the following questions must be answered:

- Does the representation contain all the essential steps for understanding the demonstration?
- Are the essential foundations written down?
- Is the demonstration correct from the logical point of view?
- Is it expressed in adequate language?

Conclusions

Familiarization with the mechanisms of demonstration used in Mathematics is a basic aspect of understanding a mathematical demonstration.

The essence of teaching proof in the subject of Mathematics lies in rationally staggering elements of intuition, induction, and conjecture, searching for the theorem, and proving it. None of these elements separately guarantees a process of teaching - learning demonstration favorable to the development of skills. Similarly, the abuse of intuitive or inductive elements hinders the learning of proof that the student needs so much.

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